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DUALITY INDUCED REFLECTIONS AND CPT

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Abstract

The linear particle-antiparticle conjugation $\mathtt{C}$ and position space reflection $\mathtt{P}$ as well as the antilinear time reflection $\mathtt{T}$ are shown to be inducable by the selfduality of representations for the operation groups $\mathbf{SU}(2)$, $\mathbf{SL}(\mathbb{C}^2)$ and $\mathbb{R}$ for spin, Lorentz transformations and time translations resp. The definition of a colour compatible linear $\mathtt{CP}$-reflection for quarks as selfduality induced is impossible since triplet and antitriplet $\mathbf{SU}(3)$-representations are not linearly equivalent.

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1 Reflections

1.1 Reflections

A reflection will be defined to be an involution of a finite dimensional vector space V

$$V \xleftrightarrow{R} V, \quad R \circ R = \mathrm{id}_V \iff R = R^{-1}$$

i.e. a realization of the parity group[2] $\mathbb{I}(2) = \{\pm 1\}$ in the V-bijections which is linear for a real space and may be linear or antilinear for a complex space

$$R(v+w) = R(v) + R(w),\ R(\alpha v) \ = \begin{cases} \alpha R(v) & \text{for } \alpha \in \mathbb{R} \text{ or } \mathbb{C} \quad \text{(linear)} \\ \overline{\alpha} R(v) & \text{for } \alpha \in \mathbb{C}, \qquad\quad \text{(antilinear)} \end{cases}$$

An antilinear reflection for a complex space $V \cong \mathbb{C}^n$ is a real linear one for its real forms $V \cong \mathbb{R}^{2n}$.

The inversion of the real numbers $\alpha \leftrightarrow -\alpha$ is the simplest nontrivial linear reflection, the canonical conjugation $\alpha \leftrightarrow \overline{\alpha}$ is the simplest nontrivial

[2] Since the parity group is used as multiplicative group, I do not use the additive notation $\mathbb{Z}_2 = \{0, 1\}$.

antilinear one being a linear one of $\mathbb{C}$ considered as real 2-dimensional space $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$.

Any (anti)linear isomorphism $\iota : V \longrightarrow W$ of two vector spaces defines an (anti)linear reflection of the direct sum $V \oplus W \stackrel{\iota \oplus \iota^{-1}}{\longleftrightarrow} V \oplus W$ which will be denoted in short also by $V \stackrel{\iota}{\longleftrightarrow} W$.

1.2 Mirrors

The fixpoints of a linear reflection $V_R^+ = \{v \mid R(v) = v\}$, i.e. the elements with even parity, in an n-dimensional space constitute a vector subspace, the mirror for the reflection R, with dimension $0 \le m \le n$ with the complement $V_R^- = \{v \mid R(v) = -v\}$, i.e. the elements with odd parity, for the direct decomposition $V = V_R^+ \oplus V_R^-$. The central reflection $R = -\mathrm{id}_V$ has the origin as a 0-dimensional mirror. Linear reflections are diagonalizable $R \cong \begin{pmatrix} \mathbf{1}_m & 0 \\ 0 & -\mathbf{1}_{n-m} \end{pmatrix}$ with $(m, n-m)$ the signature characterizing the degeneracy of ± 1 in the spectrum of R. And vice versa: Any direct decomposition $V = V^+ \oplus V^-$ defines two reflections with the mirror either V^+ or V^-.

With $(\det R)^2 = 1$ any linear reflection has either a positive or a negative orientation. Looking in the 2-dimensional bathroom mirror is formalized by the negatively oriented 3-space reflection $(x, y, z) \leftrightarrow (-x, y, z)$. The position space $\mathbb{R}^3$ reflection $\vec{x} \stackrel{-\mathbf{1}_3}{\longleftrightarrow} -\vec{x}$ with negative orientation or the Minkowski spacetime translation $\mathbb{R}^4$ reflection $x \stackrel{-\mathbf{1}_4}{\longleftrightarrow} -x$ with positive orientation are central reflections with the origins 'here' and 'here-now' as point mirrors. A space reflection $(x_0, \vec{x}) \stackrel{\mathtt{P}}{\longleftrightarrow} (x_0, -\vec{x})$ in Minkowski space or a time reflection $(x_0, \vec{x}) \stackrel{\mathtt{T}}{\longleftrightarrow} (-x_0, \vec{x})$ have both negative orientation with a 1-dimensional time and 3-dimensional position space mirror resp.

1.3 Reflections in Orthogonal Groups

A real linear reflection $R \cong \begin{pmatrix} \mathbf{1}_m & 0 \\ 0 & -\mathbf{1}_{n-m} \end{pmatrix}$ can be considered to be an element of an orthogonal group $\mathbf{O}(p, q)$ for any[3] (p, q) with $p + q = n$. A positively oriented reflection, $\det R = 1$, is is an element even of the special orthogonal groups, $R \in \mathbf{SO}(p, q)$, $p + q \ge 1$.

Orthogonal groups have discrete (semi)direct factor parity subgroups $\mathbb{I}(2)$ as seen in the simplest compact and noncompact examples

$$\begin{array}{rlll} \mathbf{O}(2) \ni & \begin{pmatrix} \epsilon\cos\alpha & \epsilon\sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, & \epsilon \in \mathbb{I}(2) = \{\pm 1\}, & \alpha \in [0, 2\pi[\\ \mathbf{O}(1,1) \ni & \epsilon' \begin{pmatrix} \epsilon\cosh\beta & \epsilon\sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix}, & \epsilon, \epsilon' \in \mathbb{I}(2), & \beta \in \mathbb{R} \end{array}$$

In general, the classes of a real orthogonal groups with respect to its special normal subgroup constitute a reflection group

$$\mathbf{O}(p, q)/\mathbf{SO}(p, q) \cong \mathbb{I}(2)$$

For real odd dimensional spaces V, e.g. for position space $\mathbb{R}^3$, one has direct products of the special groups with the central reflection group, whereas for

[3]The orthogonal signature (p, q) has nothing to do with the reflection signature (n, m).

even dimensional spaces, e.g. a Minkowski space $\mathbb{R}^4$, there arise semidirect products (denoted by $\vec{\times}$) of the special group with a reflection group which can be generated by any negatively oriented reflection

$$\mathbf{O}(p,q) \cong \left\{ \begin{array}{ll} \mathbb{I}(2) \times \mathbf{SO}(p,q), & p+q = 1,3,\dots \\ & \mathbb{I}(2) \cong \{\pm \mathrm{id}_V\} \\ \mathbb{I}(2) \vec{\times} \mathbf{SO}(p,q), & p+q = 2,4,\dots \\ & \mathbb{I}(2) \cong \{R, \mathrm{id}_V\} \text{ with } \det R = -1 \end{array} \right.$$

In the semidirect case the product is given as follows

$$(I,\Lambda) \in \mathbb{I}(2) \vec{\times} \mathbf{SO}(p,q) \Rightarrow (I_1,\Lambda_1)(I_2,\Lambda_2) = (I_1 \circ I_2, \Lambda_1 \circ I_1 \circ \Lambda_2 \circ I_1)$$

Obviously, in the semidirect case the reflection group $\mathbb{I}(2)$ is not compatible with the action of the (special) orthogonal group.

$$p+q = 2,4,\dots, \ \det R = -1 \Rightarrow [R, \mathbf{SO}(p,q)] \neq \{0\}$$

E.g. the group $\mathbf{O}(2)$ is nonabelian, or, a space reflection and a time reflection of Minkowski space is not Lorentz group $\mathbf{SO}(1,3)$ compatible.

For noncompact orthogonal groups there is another discrete reflection group: The connected subgroup G_0 (unit connection component and Lie algebra exponent) of a Lie group G is normal with a discrete quotient group G/G_0. The connected components of the full orthogonal groups are those of the special groups $\mathbf{O}_0(p,q) = \mathbf{SO}_0(p,q)$. For the compact case they are the special groups, for the noncompact ones one has two components

$$\begin{array}{ll} & \mathbf{SO}_0(n) = \mathbf{SO}(n) \\ pq \geq 1 \Rightarrow & \mathbf{SO}(p,q)/\mathbf{SO}_0(p,q) \cong \mathbb{I}(2) \end{array}$$

Summarizing: A compact orthogonal group gives rise to a reflection group $\mathbb{I}(2)$

$$\begin{array}{rl} \mathbf{O}(n) & \cong \left\{ \begin{array}{ll} \{\pm \mathbf{1}_n\} \times \mathbf{SO}(n), & n = 1,3,\dots \\ \mathbb{I}(2) \vec{\times} \mathbf{SO}(n), & n = 2,4,\dots \end{array} \right. \\ \text{with } \mathbb{I}(2) & \cong \{R, \mathbf{1}_n\}, \quad \det R = -1 \end{array}$$

a noncompact one to a reflection Klein group $\mathbb{I}(2) \times \mathbb{I}(2)$

$$\begin{array}{rl} pq \geq 1: \quad \mathbf{O}(p,q) & \cong \left\{ \begin{array}{ll} \{\pm \mathbf{1}_{p+q}\} \times [\mathbb{I}(2) \vec{\times} \mathbf{SO}_0(p,q)], & p+q = 3,5,\dots \\ \mathbb{I}(2) \vec{\times} [\{\pm \mathbf{1}_{p+q}\} \times \mathbf{SO}_0(p,q)], & p+q = 2,4,\dots \end{array} \right. \\ \text{with } \mathbb{I}(2) & \cong \{R, \mathbf{1}_n\}, \quad \det R = -1 \end{array}$$

For a noncompact $\mathbf{O}(p,q)$ with $p = 1$ the connected subgroup is the orthochronous group, compatible with the order on the vector space $V \cong \mathbb{R}^{1+q}$, e.g. for Minkowski spacetime

$$\mathbf{O}(1,3) \cong \mathbb{I}(2) \vec{\times} [\mathbb{I}(2) \times \mathbf{SO}_0(1,3)]$$

where the reflection Klein group can be generated by the central reflection $-\mathbf{1}_4$ and a position space reflection $\mathtt{P}$

$$\begin{array}{rl} \mathbb{I}(2) \times \mathbb{I}(2) \cong & \{\mathtt{P}, \mathbf{1}_4\} \times \{\pm \mathbf{1}_4\} = \{\pm \mathbf{1}_4, \mathtt{P}, \mathtt{T} = -\mathtt{P}\}, \quad [\mathbf{SO}_0(1,3), \mathtt{P}] \neq \{0\} \\ & \mathtt{P} = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbf{1}_3 \end{pmatrix}, \quad \mathtt{T} = -\mathbf{1}_4 \circ \mathtt{P} = \begin{pmatrix} -1 & 0 \\ 0 & \mathbf{1}_3 \end{pmatrix} \end{array}$$

Also the connected subgroup $\mathbf{SO}_0(p,q)$ may contain positively oriented reflections which are called continuous since they can be written as exponentials $R = e^l$ with an element of the orthogonal Lie algebra[4], $l \in \log \mathbf{SO}_0(p,q)$. E.g the central reflections $-\mathbf{1}_{2n} \in \mathbf{SO}(2n)$ in even dimensional Euclidean spaces, e.g. in the Euclidean 2-plane. A negatively oriented reflection R of a space V can be embedded as a reflection $R \oplus S$ with any orientation of a strictly higher dimensional space $V \oplus W$

$$\begin{array}{rl} V \xleftrightarrow{R} V, & \det R = -1 \\ V \oplus W \xleftrightarrow{R \oplus S} V \oplus W, & \det(R \oplus S) = -\det S \end{array}$$

where, for compact orthogonal groups on V and $V \oplus W$, a reflection $R \oplus S$ with $\det S = -1$ is a continuous reflection, i.e. a rotation. There are the familiar examples[2] for $\mathbf{O}(n) \hookrightarrow \mathbf{SO}(n+1)$: Two letter noodles in L-form, lying with opposite helicity on the kitchen table, can be 3-space rotated into each other, or, a left and a right handed glove are identical up to Euclidean 4-space rotations. The embedding of the central position space reflection into Minkowski spacetime can go into a positively or negatively oriented reflection which are both not continuous, i.e. they are in the discrete Klein reflection group

$$-\mathbf{1}_3 \hookrightarrow \begin{pmatrix} \pm 1 & 0 \\ 0 & -\mathbf{1}_3 \end{pmatrix}, \quad \{\mathtt{P}, -\mathbf{1}_4\} \subset \mathbf{O}(1,3)/\mathbf{SO}_0(1,3)$$

2 Reflections for Spinors

The doubly connected groups $\mathbf{SO}(3)$ and $\mathbf{SO}_0(1,3)$ can be complex represented via their simply connected covering groups $\mathbf{SU}(2)$ and[5] $\mathbf{SL}(\mathbb{C}^2)$ resp.

$$\mathbf{SO}(3) \cong \mathbf{SU}(2)/\{\pm\mathbf{1}_2\}, \quad \mathbf{SO}_0(1,3) \cong \mathbf{SL}(\mathbb{C}^2)/\{\pm\mathbf{1}_2\},$$

The reflection group $\{\pm\mathbf{1}_2\}$ for the $\mathbf{SO}(3)$-classes in $\mathbf{SU}(2)$ and the $\mathbf{SO}_0(1,3)$-classes in $\mathbf{SL}(\mathbb{C}^2)$ contains the continuous central $\mathbb{C}^2$-reflection $-\mathbf{1}_2 = e^{i\pi\sigma_3} \in \mathbf{SU}(2)$.

2.1 The Pauli Spinor Reflection

The fundamental defining $\mathbf{SU}(2)$-representation for the rotations acts on Pauli spinors $W \cong \mathbb{C}^2$

$$u = e^{i\vec{\alpha}\vec{\sigma}} \in \mathbf{SU}(2) \text{ (Pauli matrices } \vec{\sigma})$$

They have an invariant antisymmetric bilinear form (spinor 'metric')

$$\epsilon : W \times W \longrightarrow \mathbb{C}, \quad \epsilon(\psi^A, \psi^B) = \epsilon^{AB} = -\epsilon^{BA}, \quad A, B = 1, 2$$

[4] $\log G$ denotes the Lie algebra of the Lie group G.

[5] Throughout this paper the group $\mathbf{SL}(\mathbb{C}^2)$ is used as *real* 6-dimensional Lie group.

which defines an isomorphism with the dual[6] space $W^T \cong \mathbb{C}^2$ is compatible with the $\mathbf{SU}(2)$-action - on the dual space as dual representation $\check{u}$ (inverse transposed)

$$\begin{tikzcd} W \arrow[r, "u"] \arrow[d, "\epsilon"'] & W \arrow[d, "\epsilon"] \\ W^T \arrow[r, "\check{u}"'] & W^T \end{tikzcd}, \quad \begin{array}{rl} \psi^A & \leftrightarrow \epsilon^{AB}\psi^\star_B \\ \check{u} & = u^{-1T} = u^{\star-1} = \overline{u} = (e^{-i\vec{\alpha}\vec{\sigma}})^T \\ u & = \epsilon^{-1} \circ \check{u} \circ \epsilon \\ -\vec{\sigma} & = \epsilon^{-1} \circ \vec{\sigma}^T \circ \epsilon \end{array}$$

ϵ connects the two Pauli representations with reflected transformations of the spin Lie algebra $\log \mathbf{SU}(2)$, i.e. it defines a central reflection for the three compact rotation parameters $\vec{\alpha}$

$$\begin{array}{c} e^{i\vec{\alpha}\vec{\sigma}} \stackrel{\epsilon}{\longleftrightarrow} (e^{-i\vec{\alpha}\vec{\sigma}})^T \\ i\vec{\alpha}\vec{\sigma} \in \log \mathbf{SU}(2) \cong \mathbb{R}^3, \quad \vec{\alpha} \stackrel{\epsilon}{\longleftrightarrow} -\vec{\alpha} \end{array}$$

and will be called the *Pauli spinor reflection*

$$W \stackrel{\epsilon}{\longleftrightarrow} W^T, \quad \psi^A \leftrightarrow \epsilon^{AB}\psi^\star_B, \quad [\epsilon, \mathbf{SU}(2)] = \{0\}$$

The mathematical structure of selfduality as a reflection generating mechnism is given in the appendix.

2.2 Reflections C and P for Weyl Spinors

The two fundamental $\mathbf{SL}(\mathbb{C}^2)$-representations for the Lorentz group are the the left and right handed Weyl representation on $W_L, W_R \cong \mathbb{C}^2$ with the dual representations on the linear forms $W^T_{L,R}$

$$\begin{array}{rlrl} \text{left: } \lambda & = e^{(i\vec{\alpha}+\vec{\beta})\vec{\sigma}} \in \mathbf{SL}(\mathbb{C}^2), & \text{right: } \hat{\lambda} & = \lambda^{-1\star} = e^{(i\vec{\alpha}-\vec{\beta})\vec{\sigma}} \\ \text{left dual: } \check{\lambda} = \lambda^{-1T} & = [e^{(-i\vec{\alpha}-\vec{\beta})\vec{\sigma}}]^T, & \text{right dual: } \lambda^{T\star} = \overline{\lambda} & = [e^{(-i\vec{\alpha}+\vec{\beta})\vec{\sigma}}]^T \end{array}$$

The Weyl representations with dual bases in the conventional notations with dotted and undotted indices[7]

$$\begin{array}{rlrl} \text{left: } & \mathrm{l}^A \in W_L \cong \mathbb{C}^2, & \text{right: } & \mathrm{r}^{\dot{A}} \in W_R \cong \mathbb{C}^2 \\ \text{left dual: } & \mathrm{r}^\star_A \in W^T_L \cong \mathbb{C}^2, & \text{right dual: } & \mathrm{l}^\star_{\dot{A}} \in W^T_R \cong \mathbb{C}^2 \end{array}$$

[6] The linear forms V^T of a vector space V define the dual product $V^T \times V \longrightarrow \mathbb{C}$ by $\langle \omega, v \rangle = \omega(v)$ and dual bases by $\langle \check{e}_j, e^k \rangle = \delta^k_j$. Transposed mappings $f : V \longrightarrow W$ are denoted by $f^T : W^T \longrightarrow V^T$ with $\langle f^T(\omega), v \rangle = \langle \omega, f(v) \rangle$.

[7] The usual strange looking crossover association of the letters $\mathrm{l}^\star$ and $\mathrm{r}^\star$ for right and left handed dual spinors resp. will be discussed later.

are selfdual with the $\mathbf{SL}(\mathbb{C}^2)$-invariant volume form on $\mathbb{C}^2$, i.e. the dual isomorphisms are Lorentz compatible

$$\begin{array}{ccc} W_L & \stackrel{\lambda}{\longrightarrow} & W_L \\ \epsilon_L \downarrow & & \downarrow \epsilon_L \\ W_L^T & \stackrel{\longrightarrow}{\check{\lambda}} & W_L^T \end{array}, \qquad \begin{array}{ccc} W_R & \stackrel{\hat{\lambda}}{\longrightarrow} & W_R \\ \epsilon_R \downarrow & & \downarrow \epsilon_R \\ W_R^T & \stackrel{\longrightarrow}{\overline{\lambda}} & W_R^T \end{array}$$

For the Lorentz group the spinor 'metric' will prove to be related to the *particle-antiparticle conjugation*, and will be called *Weyl spinor reflection*, denoted by $\mathtt{C} \in \{\epsilon_L, \epsilon_R\}$

$$\begin{array}{ll} W_L \stackrel{\mathtt{C}}{\longleftrightarrow} W_L^T, & \mathrm{l}^A \leftrightarrow \epsilon^{AB}\mathrm{r}^\star_B \\ W_R \stackrel{\mathtt{C}}{\longleftrightarrow} W_R^T, & \mathrm{r}^{\dot{A}} \leftrightarrow \epsilon^{\dot{A}\dot{B}}\mathrm{l}^\star_{\dot{B}} \end{array}$$

There exist isomorphisms δ between left and right handed Weyl spinors, compatible with the spin group action, however not with the Lorentz group $\mathbf{SL}(\mathbb{C}^2)$

$$\begin{array}{ccc} W_L & \stackrel{u_L}{\longrightarrow} & W_L \\ \delta \downarrow & & \downarrow \delta \\ W_R & \stackrel{\longrightarrow}{u_R} & W_R \end{array}, \qquad \begin{array}{l} u_{L,R} = e^{i\vec{\alpha}\vec{\sigma}} \in \mathbf{SU}(2) \\ \mathrm{l}^A \leftrightarrow \delta^A_{\dot{A}}\mathrm{r}^{\dot{A}} \end{array}$$

They connect representations with a reflected boost transformation, i.e. they define a central reflection for the three noncompact boost parameters $\vec{\beta}$

$$\begin{array}{c} e^{(i\vec{\alpha}+\vec{\beta})\vec{\sigma}} \stackrel{\delta}{\longleftrightarrow} e^{(i\vec{\alpha}-\vec{\beta})\vec{\sigma}} \\ \vec{\sigma}\vec{\beta} \in \log \mathbf{SL}(\mathbb{C}^2)/\log \mathbf{SU}(2) \cong \mathbb{R}^3, \quad \vec{\beta} \stackrel{\delta}{\longleftrightarrow} -\vec{\beta} \end{array}$$

These isomorphisms induce nontrivial reflections of the Dirac spinors $\Psi \in W_L \oplus W_R \cong \mathbb{C}^4$

$$\Psi = \begin{pmatrix} \mathrm{l}^A \\ \mathrm{r}^{\dot{A}} \end{pmatrix} \stackrel{\delta}{\longleftrightarrow} \begin{pmatrix} 0 & \delta^A_{\dot{B}} \\ \delta^{\dot{A}}_B & 0 \end{pmatrix} \begin{pmatrix} \mathrm{l}^B \\ \mathrm{r}^{\dot{B}} \end{pmatrix} = \gamma^0 \Psi$$

with the chiral representation of the Dirac matrices

$$\gamma^j = \begin{pmatrix} 0 & \sigma^j \\ \check{\sigma}^j & 0 \end{pmatrix}, \quad \sigma^j = (\mathbf{1}_2, \vec{\sigma}), \quad \check{\sigma}^j = (\mathbf{1}_2, -\vec{\sigma})$$

and will be called *Weyl spinor boost reflections* $\mathtt{P} = \delta$, later used for the central *position space reflection* representation

$$\begin{array}{ll} W_L \stackrel{\mathtt{P}}{\longleftrightarrow} W_R, & \mathrm{l}^A \leftrightarrow \delta^A_{\dot{A}}\mathrm{r}^{\dot{A}} \\ W_L^T \stackrel{\mathtt{P}}{\longleftrightarrow} W_R^T, & \mathrm{r}^\star_A \leftrightarrow \delta^{\dot{A}}_A\mathrm{l}^\star_{\dot{A}} \end{array}$$

Therewith all four Weyl spinor spaces are connected to each other by linear reflections

$$\begin{array}{ccccc} & & \mathtt{P} & & \\ & W_L & \longleftrightarrow & W_R & \\ \mathtt{C} & \updownarrow & & \updownarrow & \mathtt{C}\,, \\ & W_L^T & \longleftrightarrow & W_R^T & \\ & & \mathtt{P} & & \end{array} \quad \begin{array}{l} [\mathtt{P}, \mathbf{SL}(\mathbb{C}^2)] \neq \{0\}, \quad [\mathtt{P}, \mathbf{SU}(2)] = \{0\} \\ [\mathtt{C}, \mathbf{SL}(\mathbb{C}^2)] = \{0\} \end{array}$$

3 Time Reflection

The time representations define the antilinear reflection $\mathtt{T}$ for time translation. The different duality with respect to $\mathbf{SL}(\mathbb{C}^2)$ and Lorentz group representations, on the one side, and time representations, on the other side, leads to the nontrivial $\mathtt{C}, \mathtt{P}, \mathtt{T}$ cooperation.

3.1 Reflection $\mathtt{T}$ of Time Translations

The irreducible time representations, familiar from the quantum mechanical harmonic oscillator with time action eigenvalue (frequency) ω, with their duals (inverse transposed) are complex 1-dimensional

$$t \longmapsto e^{i\omega t} \in \mathbf{GL}(U), \quad t \longmapsto e^{-i\omega t} \in \mathbf{GL}(U^T), \quad U \cong \mathbb{C} \cong U^T$$

They are selfdual (equivalent) with an antilinear dual isomorphism which is the $\mathbf{U}(1)$-conjugation for a dual basis $\mathrm{u} \in U$, $\mathrm{u}^\star \in U^T$

$$\begin{array}{ccccc} & & e^{i\omega t} & & \\ & U & \longrightarrow & U & \\ \star & \downarrow & & \downarrow & \star\,, \\ & U^T & \longrightarrow & U^T & \\ & & e^{-i\omega t} & & \end{array} \qquad \mathrm{u} \leftrightarrow \mathrm{u}^\star$$

The antilinear isomorphism $\star$ defines a scalar product which gives rise to the quantum mechanical probability amplitudes (Fock state for the harmonic oscillator)

$$U \times U \longrightarrow \mathbb{C}, \quad \langle \mathrm{u}|\mathrm{u}\rangle = \langle \mathrm{u}^\star, \mathrm{u}\rangle = 1$$

and defines the *time reflection* $\mathtt{T} = \star$ for the time translations

$$e^{i\omega t} \overset{\mathtt{T}}{\longleftrightarrow} e^{-i\omega t}, \quad t \overset{\mathtt{T}}{\longleftrightarrow} -t$$

3.2 Lorentz Duality versus Time Duality

As anticipated in the conventional, on first sight strange looking dual Weyl spinor notation, e.g. $\mathrm{l} \in W_L$ and $\mathrm{l}^\star \in W_R^T$, the Weyl spinor spaces $W_{L,R}^T$ with the dual left and right handed $\mathbf{SL}(\mathbb{C}^2)$-representations are not the spaces with the dual time respesentations as exemplified in the harmonic analysis of the left and right handed components in a Dirac field

$$\begin{array}{rll}
\mathrm{l}^A(x) & = \int \frac{d^3q}{(2\pi)^3} \quad s(\frac{q}{m})^A_C & \frac{e^{xiq}\mathrm{u}^C(\vec q)+e^{-xiq}\mathrm{a}^{\star C}(\vec q)}{\sqrt{2}}\\
\mathrm{l}^\star_{\dot A}(x) & = \int \frac{d^3q}{(2\pi)^3} \quad s^\star(\frac{q}{m})^C_{\dot A} & \frac{e^{-xiq}\mathrm{u}^\star_C(\vec q)+e^{xiq}\mathrm{a}_C(\vec q)}{\sqrt{2}}\\
\mathrm{r}^{\dot A}(x) & = \int \frac{d^3q}{(2\pi)^3} \quad s^{\star -1}(\frac{q}{m})^{\dot A}_C & \frac{e^{xiq}\mathrm{u}^C(\vec q)-e^{-xiq}\mathrm{a}^{\star C}(\vec q)}{\sqrt{2}}\\
\mathrm{r}^\star_A(x) & = \int \frac{d^3q}{(2\pi)^3} \quad s^{-1}(\frac{q}{m})^C_A & \frac{e^{-xiq}\mathrm{u}^\star_C(\vec q)-e^{xiq}\mathrm{a}_C(\vec q)}{\sqrt{2}}
\end{array}$$
$$s(\tfrac{q}{m}) = \sqrt{\tfrac{q_0+m}{2m}}(\mathbf{1} + \tfrac{\vec\sigma\vec q}{q_0+m}), \quad q = (q_0, \vec q), \quad q_0 = \sqrt{m^2+\vec q^2}$$

Here, $s(\frac{q}{m}) \in \mathbf{SL}(\mathbb{C}^2)$ is the Weyl representation of the boost from the rest system of the particle to a frame moving with velocity $\frac{\vec q}{q_0}$ (solution of the Dirac equation), u^C and a_C are the creation operators for particle and antiparticles with spin $\frac{1}{2}$ and opposite charge number ± 1 and 3rd spin direction, e.g. for electron and positron, $\mathrm{u}^\star_C$ and $\mathrm{a}^{\star C}$ are the corresponding annihilation operators.

$\star$ denotes the time representation dual $U \leftrightarrow U^\star$, and T the Lorentz representation dual $W \leftrightarrow W^T$ (with spinor indices up and down), i.e. for the four types of Weyl spinors

$$\begin{array}{cccc}
 & & \text{time dual} & & \\
 & \mathrm{l}^A \in W_L & \longleftrightarrow & \mathrm{l}^\star_{\dot A} \in W_R^T = W_L^\star & \\
\text{Lorentz dual} & \updownarrow & & \updownarrow & \text{Lorentz dual}\\
 & \mathrm{r}^\star_A \in W_L^T = W_R^\star & \longleftrightarrow & \mathrm{r}^{\dot A} \in W_R & \\
 & & \text{time dual} & &
\end{array}$$

Time representation duality does not coincide with Lorentz group representation duality.

The antilinear time reflection ($\mathbf{U}(1)$-conjugation) $\mathtt{T} = \star$ is compatible with the action of the little group $\mathbf{SU}(2)$, not with the full Lorentz group

$$\left.\begin{array}{l}
W_L \xleftrightarrow{\mathtt{T}} W_R^T, \quad \mathrm{l}^A \leftrightarrow \delta^{A\dot A}\mathrm{l}^\star_{\dot A}\\
W_R \xleftrightarrow{\mathtt{T}} W_L^T, \quad \mathrm{r}^{\dot A} \leftrightarrow \delta^{\dot A A}\mathrm{r}^\star_A
\end{array}\right\}, \quad [\mathtt{T}, \mathbf{SL}(\mathbb{C}^2)] \neq 0, \quad [\mathtt{T}, \mathbf{SU}(2)] = 0$$

3.3 The Cooperation of $\mathtt{C}, \mathtt{P}, \mathtt{T}$ in the Lorentz Group

It is useful to summarize the action of the linear Weyl spinor reflections $\mathtt{C}$ (particle-antiparticle conjugation) and $\mathtt{P}$ (position space central reflection) and the antilinear time reflection $\mathtt{T}$ in the two types of commuting diagrams

$$\begin{tikzcd} W_L \arrow[r, leftrightarrow, "\mathtt{P}"] \arrow[d, leftrightarrow, "\mathtt{C}"'] & W_R \arrow[d, leftrightarrow, "\mathtt{C}"] \\ W_L^T \arrow[r, leftrightarrow, "\mathtt{P}"'] & W_R^T \end{tikzcd} \quad \text{with} \quad \begin{tikzcd} \mathrm{l}^A \arrow[r, leftrightarrow, "\mathtt{P}"] \arrow[d, leftrightarrow, "\mathtt{C}"'] & \delta^A_{\dot A}\mathrm{r}^{\dot A} \arrow[d, leftrightarrow, "\mathtt{C}"] \\ \epsilon^{AB}\mathrm{r}^\star_B \arrow[r, leftrightarrow, "\mathtt{P}"'] & \delta^A_{\dot A}\epsilon^{\dot A\dot B}\mathrm{l}^\star_{\dot B} \end{tikzcd}$$

$$\begin{tikzcd} W_L \arrow[r, leftrightarrow, "\mathtt{T}"] \arrow[d, leftrightarrow, "\mathtt{C}"'] & W_R^T \arrow[d, leftrightarrow, "\mathtt{C}"] \\ W_L^T \arrow[r, leftrightarrow, "\mathtt{T}"'] & W_R \end{tikzcd} \quad \text{with} \quad \begin{tikzcd} \mathrm{l}^A \arrow[r, leftrightarrow, "\mathtt{T}"] \arrow[d, leftrightarrow, "\mathtt{C}"'] & \delta^{A\dot B}\mathrm{l}^\star_{\dot B} \arrow[d, leftrightarrow, "\mathtt{C}"] \\ \epsilon^{AB}\mathrm{r}^\star_B \arrow[r, leftrightarrow, "\mathtt{T}"'] & \delta^{A\dot B}\epsilon_{\dot B\dot A}\mathrm{r}^{\dot A} \end{tikzcd}$$

with the compatibilities

$$\begin{array}{l} [\mathtt{C}, \mathbf{SL}(\mathbb{C}^2)] = \{0\}, \quad [\mathtt{P} \text{ and } \mathtt{T}, \mathbf{SL}(\mathbb{C}^2)] \neq \{0\}, \quad [\mathtt{P} \text{ and } \mathtt{T}, \mathbf{SU}(2)] = \{0\} \\ [\mathtt{C}, \mathtt{P}] = 0, \quad [\mathtt{C}, \mathtt{T}] = 0, \quad [\mathtt{P}, \mathtt{T}] = 0 \end{array}$$

The product $\mathtt{CPT}$ is an antilinear reflection of each Weyl spinor space, e.g. for the left handed spinors

$$W_L \xleftrightarrow{\mathtt{CPT}} W_L, \quad \mathrm{l}^A \leftrightarrow \delta^A_{\dot A}\epsilon^{\dot A\dot B}\delta_{\dot B B}\mathrm{l}^B$$

involving an element of the group $\mathbf{SL}(\mathbb{C}^2)$, even of $\mathbf{SU}(2)$

$$\mathtt{CPT} \sim \delta^A_{\dot A}\epsilon^{\dot A\dot B}\delta_{\dot B B} = u^A_B \cong \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = e^{i\frac{\pi}{2}\sigma_2} \in \mathbf{SU}(2) \subset \mathbf{SL}(\mathbb{C}^2)$$

This element gives - in the used basis - for the Lorentz group a π-rotation around the 2nd axis in position space, i.e. a continuous reflection

$$\mathbf{SU}(2) \ni e^{i\frac{\pi}{2}\sigma_2} \longmapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathbf{SO}(3), \quad (x, y, z) \leftrightarrow (-x, y, -z)$$

The fact that the antilinear $\mathtt{CPT}$-reflection is - up to a number conjugation (indicated by overlining) - an element of $\mathbf{SL}(\mathbb{C}^2)$, covering the connected Lorentz group $\mathbf{SO}_0(1,3)$, is decisive for the proof of the well known $\mathtt{CPT}$-theorem[4, 3]

$$\overline{\mathtt{CPT}} \in \mathbf{SL}(\mathbb{C}^2)$$

4 Spinor Induced Reflections

The linear spinor reflections ϵ for Pauli spinors and $\mathtt{C}, \mathtt{P}$ for Weyl spinors are inducable on all irreducible finite dimensional representations of $\mathbf{SU}(2)$ and

$\mathbf{SL}(\mathbb{C}^2)$ with their adjoint groups $\mathbf{SO}(3)$ and $\mathbf{SO}_0(1,3)$ resp. via the general procedure: Given the group G action on two vector spaces its tensor product representation reads

$$G \times (V_1 \otimes V_2) \longrightarrow V_1 \otimes V_2, \quad g \bullet (v_1 \otimes v_2) = (g \bullet v_1) \otimes (g \bullet v_2)$$

A realization of the simple reflection group $\mathbb{I}(2) = \{\pm 1\}$ is either faithful or trivial.

4.1 Spinor Induced Reflection of Position Space

The reflection $W \stackrel{\epsilon}{\longleftrightarrow} W^T$ for a Pauli spinor space $W \cong \mathbb{C}^2$ induces the central reflection of position space whose elements come - in the Pauli representation of position space - as traceless hermitian (2×2)-matrices

$$\vec{x} : W \longrightarrow W, \quad \operatorname{tr} \vec{x} = 0, \quad \vec{x} = \vec{x}^\star = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}$$

i.e. as elements[8] of the tensor product $W \otimes W^T$ with the induced ϵ-reflection

$$-\vec{\sigma} = \epsilon^{-1} \circ \vec{\sigma}^T \circ \epsilon \ \Rightarrow \ \vec{x} \stackrel{\epsilon}{\longleftrightarrow} \epsilon^{-1} \circ \vec{x}^T \circ \epsilon = -\vec{x}$$

In the Cartan representation the Minkowski spacetime translations are hermitian mappings from right handed to left handed spinors

$$x : W_R \longrightarrow W_L, \quad x = x^\star = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}$$

i.e. tensors in the product $W_L \otimes W_R^T$. The linear $\mathtt{CP}$-reflection for Weyl spinors

$$W_L \stackrel{\mathtt{CP}}{\longleftrightarrow} W_R^T, \quad W_R \stackrel{\mathtt{CP}}{\longleftrightarrow} W_L^T$$

induces the position space reflection of Minkowski spacetime

$$\begin{array}{l} \sigma^j = (\mathbf{1}_2, \vec{\sigma}), \quad \epsilon^{-1} \circ (\sigma^j)^T \circ \epsilon = \sigma_j = (\mathbf{1}_2 - \vec{\sigma}) \\ x \cong (x_0, \vec{x}) \stackrel{\mathtt{CP}}{\longleftrightarrow} \epsilon^{-1} \circ x^T \circ \epsilon = \begin{pmatrix} x_0 - x_3 & -x_1 + ix_2 \\ -x_1 - ix_2 & x_0 + x_3 \end{pmatrix} \cong (x_0, -\vec{x}) \end{array}$$

4.2 Induced Reflections of Spin Representation Spaces

All irreducible complex representations of the spin group $\mathbf{SU}(2)$ with $2J = 0, 1, 2, \dots$ have an invariant bilinear form arising as a symmetric tensor product of the antisymmetric spinor 'metric' ϵ. The bilinear form is given for the irreducible representation $[2J] \cong \bigvee^{2J} u$ on the vector space $\bigvee^{2J} W \cong \mathbb{C}^{2J+1}$ by the corresponding totally symmetric[9] power and is antisymmetric for halfinteger spin and symmetric for integer spin

$$\epsilon^{2J} = \bigvee^{2J} \epsilon, \quad \epsilon^{2J}(v, w) = \begin{cases} +\epsilon^{2J}(w, v), & 2J = 0, 2, 4 \dots \\ -\epsilon^{2J}(w, v), & 2J = 1, 3, \dots \end{cases}$$

[8] The linear mappings $\{V \longrightarrow W\}$ for finite dimensional vector spaces are naturally isomorphic to the tensor product $W \otimes V^T$ with the linear V-forms V^T.

[9] $\bigvee$ and $\bigwedge$ denotes symmetrized and antisymmetrized tensor products.

The complex representation spaces for integer spin $J = 0, 1, \ldots$, acted upon faithfully only with the special rotations $\mathbf{SO}(3) \cong \mathbf{SU}(2)/\{\pm\mathbf{1}_2\}$, are direct sums of two irreducible real $\mathbf{SO}(3)$-representation spaces $\mathbb{R}^{2J+1}$ where the invariant bilinear form is symmetric and definite, e.g. the negative definite Killing form $-\mathbf{1}_3$ for the adjoint representation $[2] \cong u \vee u$ on $\mathbb{R}^3$.

The Pauli spinor reflection induces the reflections for the irreducible spin representation spaces

$$V \cong \bigvee^{2J} W \cong \mathbb{C}^{2J+1} : \quad V \stackrel{\epsilon^{2J}}{\longleftrightarrow} V^T$$

For integer spin (odd dimensional representation spaces) the two real subspaces with irreducible real $\mathbf{SO}(3)$-representation come with a trivial $-\mathbf{1}_3 \longmapsto \mathbf{1}_{2J+1}$ and a faithful $-\mathbf{1}_3 \longmapsto -\mathbf{1}_{2J+1} \in \mathbf{O}(2J+1)/\mathbf{SO}(2J+1)$ representation of the central position space reflection, as seen in the diagonalization of the induced reflection

$$\begin{pmatrix} 0 & \epsilon^{2J} \\ [\epsilon^{2J}]^{-1} & 0 \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & J = 0 \\ \begin{pmatrix} 0 & \epsilon \\ \epsilon^{-1} & 0 \end{pmatrix}, & J = \frac{1}{2} \\ \begin{pmatrix} 0 & -\mathbf{1}_3 \\ -\mathbf{1}_3 & 0 \end{pmatrix} \cong \begin{pmatrix} \mathbf{1}_3 & 0 \\ 0 & -\mathbf{1}_3 \end{pmatrix}, & J = 1 \\ \text{etc.} \end{cases}$$

The decomposition for the integer spin representation spaces uses symmetric and antisymmetric tensor products as illustrated for the scalar and vector spin representation with a Pauli spinor basis

$$\begin{array}{ll} W \stackrel{\epsilon}{\longleftrightarrow} W^T, & \psi^A \leftrightarrow \epsilon^{AB}\psi^\star_B, \quad J = \frac{1}{2} \\ W^T \otimes W \stackrel{\epsilon}{\longleftrightarrow} W \otimes W^T, & \begin{cases} \psi^\star_A \otimes \psi^A \leftrightarrow \psi^A \otimes \psi^\star_A, & J = 0 \\ \vec{\sigma}^A_B \psi^\star_A \otimes \psi^B \leftrightarrow -\vec{\sigma}^A_B \psi^B \otimes \psi^\star_A, & J = 1 \end{cases} \end{array}$$

Writing for the tensor (anti)commutator $[a, b]_\epsilon = a \otimes b + \epsilon b \otimes a$ with $\epsilon = \pm 1$ one has in both cases one trivial and one faithful reflection representation

$$\begin{array}{rcll} [\psi^\star_A, \psi^A]_\epsilon & \leftrightarrow & \epsilon[\psi^\star_A, \psi^A]_\epsilon & J = 0 \\ [\psi^\star_A \vec{\sigma}^A_B, \psi^B]_\epsilon & \leftrightarrow & -\epsilon[\psi^\star_A \vec{\sigma}^A_B, \psi^B]_\epsilon, & J = 1 \end{array}$$

4.3 Induced Reflections of Lorentz Group Representation Spaces

The generating structure of the two Weyl representations induces $\mathtt{C}, \mathtt{P}$-reflections of $\mathbf{SL}(\mathbb{C}^2)$-representations spaces.

The complex finite dimensional irreducible representations of the group $\mathbf{SL}(\mathbb{C}^2)$ are characterized by two spins $[2L|2R]$ with integer and halfinteger

$L, R = 0, \frac{1}{2}, 1, \ldots$. They are equivalent to the totally symmetric products of the left and right handed Weyl representations

$$\begin{array}{l}\text{Weyl left: } [1|0] = \lambda = e^{(i\vec{\alpha}+\vec{\beta})\vec{\sigma}}, \;\; \text{Weyl right: } [0|1] = \hat{\lambda} = e^{(i\vec{\alpha}-\vec{\beta})\vec{\sigma}} \\ [2L|2R] \cong \bigvee^{2L}\lambda \otimes \bigvee^{2R}\hat{\lambda} \text{ acting on } V \cong \bigvee^{2L} W_L \otimes \bigvee^{2R} W_R \cong \mathbb{C}^{(2L+1)(2R+1)}\end{array}$$

$[2L|2R]$ and $[2R|2L]$ are equivalent with respect to the subgroup $\mathbf{SU}(2)$-representations. The induced reflections are given by the correponding products of the Weyl spinor reflections.

The real representation spaces for the Lorentz group $\mathbf{SO}_0(1, 3)$ are characterized by integer spin

$$L + R = 0, 1, 2, \ldots$$

They are all generated by the Minkowski representation $[1|1] \cong \lambda \otimes \overline{\lambda}$ where the complex 4-dimensional representation space is decomposable into two real 4-dimensional ones, a hermitian and an antihermitian tensor

$$\mathbb{C}^4 \cong W_L \otimes W_R^T \ni \mathrm{l} \otimes \mathrm{l}^\star = z = x + i\alpha \in \mathbb{R}^4 \;\; \oplus \;\; i\mathbb{R}^4$$

With Weyl spinor bases the induced linear reflections for the Minkowski representation look as follows (with $\sigma^j = (\mathbf{1}_2, \vec{\sigma}) = \check{\sigma}_j$ and $\sigma_j = (\mathbf{1}_2, -\vec{\sigma}) = \check{\sigma}^j$)

$$\begin{array}{l}\sigma^j \xleftrightarrow{\mathtt{P}} \check{\sigma}_j^T, \quad \mathrm{l}^\star\sigma^j\mathrm{l} \xleftrightarrow{\mathtt{P}} \mathrm{r}^\star\check{\sigma}_j^T\mathrm{r} \\ \sigma_j \xleftrightarrow{\mathtt{C}} \check{\sigma}_j^T, \quad \mathrm{l}^\star\sigma_j\mathrm{l} \xleftrightarrow{\mathtt{C}} \mathrm{r}\check{\sigma}_j^T\mathrm{r}^\star \\ \sigma^j \xleftrightarrow{\mathtt{CP}} \sigma_j^T, \quad \mathrm{l}^\star\sigma^j\mathrm{l} \xleftrightarrow{\mathtt{CP}} \mathrm{l}\sigma_j^T\mathrm{l}^\star, \quad \mathrm{r}^\star\check{\sigma}^j\mathrm{r} \xleftrightarrow{\mathtt{CP}} \mathrm{r}\check{\sigma}_j^T\mathrm{r}^\star\end{array}$$

and can be arranged in combinations of definite parity, e.g. for P with Dirac spinors in a vector $\overline{\Psi}\gamma^j\Psi$ and an axial vector $\overline{\Psi}\gamma^j\gamma_5\Psi$. The antilinear time reflection has to change in addition the order in the product

$$\sigma^j \xleftrightarrow{\mathtt{T}} \sigma_j, \quad \mathrm{l}^\star\sigma^j\mathrm{l} \xleftrightarrow{\mathtt{T}} \mathrm{l}^\star\sigma_j\mathrm{l}, \quad \mathrm{r}^\star\check{\sigma}^j\mathrm{r} \xleftrightarrow{\mathtt{T}} \mathrm{r}^\star\check{\sigma}_j\mathrm{r}$$

4.4 Reflections of Spacetime Fields

A field Φ is a mapping from position space $\mathbb{R}^3$ or, as relativistic field, from Minkowski spacetime $\mathbb{R}^4$ with values in a complex vector space V with the action of a group G both on space(time) and on V. This defines the action of the group on the field $\Phi \longmapsto g \bullet \Phi = {}_g\Phi$ by the commutativity of the diagram

$$\begin{array}{ccccc} & \mathbb{R}^3, \mathbb{R}^4 & \xrightarrow{O(g)} & \mathbb{R}^3, \mathbb{R}^4 & \\ \mathbf{\Phi} & \downarrow & & \downarrow & {}_g\mathbf{\Phi}\,, \\ & V & \xrightarrow[D(g)]{} & V & \end{array} \qquad \begin{array}{l} {}_g\mathbf{\Phi}(x) = D(g)\mathbf{\Phi}(O(g^{-1}).x) \\ \text{for } g \in G \end{array}$$

For position space the external action group is the Euclidean group $\mathbf{O}(3) \vec{\times} \mathrm{I\!R}^3$, for Minkowski spacetime the Poincaré group $\mathbf{O}(1,3) \vec{\times} \mathrm{I\!R}^4$. The value space may have additional interal action groups, e.g. $\mathbf{U}(1)$, $\mathbf{SU}(2)$ and $\mathbf{SU}(3)$ hypercharge, isospin and colour resp. in the standard model for quark and lepton fields.

For Pauli spinor fields on position space the $\mathbf{O}(3)$-action has a direct $\mathbf{SU}(2)$-factor and a reflection factor $\mathrm{I\!I}(2)$

$$\psi : \mathrm{I\!R}^3 \longrightarrow W \cong \mathbb{C}^2, \quad \begin{cases} {}_u\psi(\vec{x}) = D(u)\psi(O(u^{-1}).\vec{x}), & u \in \mathbf{SU}(2),\ O(u) \in \mathbf{SO}(3) \\ \psi^A(\vec{x}) \stackrel{\epsilon}{\longleftrightarrow} \epsilon^{AB}\psi^\star_B(-\vec{x}), & \text{position reflection } \mathrm{I\!I}(2) \end{cases}$$

Spacetime fields have the Lorentz group behaviour

$${}_\lambda\Phi(x) = D(\lambda).\Phi(O(\lambda^{-1}).x), \quad \lambda \in \mathbf{SL}(\mathbb{C}^2), \quad O(\lambda) \in \mathbf{SO}_0(1,3)$$

The antilinear time reflection uses the conjugation to the time dual field

$$\Phi(x_0,\vec{x}) \stackrel{\mathtt{T}}{\longleftrightarrow} \Phi^\star(-x_0,\vec{x})$$

The reflections for Weyl spinor fields on Minkowski spacetime are

$$\begin{array}{cccccc} \mathrm{l}^A & (x_0,\vec{x}) & \stackrel{\mathtt{P}}{\longleftrightarrow} & \delta^A_{\dot{A}}\mathrm{r}^{\dot{A}} & (x_0,-\vec{x}) \\ (\mathrm{l}^A,\mathrm{r}^{\dot{A}}) & (x_0,\vec{x}) & \stackrel{\mathtt{C}}{\longleftrightarrow} & (\epsilon^{AB}\mathrm{r}^\star_B, \epsilon^{\dot{A}\dot{B}}\mathrm{l}^\star_{\dot{B}}) & (x_0,\vec{x}) \\ (\mathrm{l}^A,\mathrm{r}^{\dot{A}}) & (x_0,\vec{x}) & \stackrel{\mathtt{CP}}{\longleftrightarrow} & (\delta^A_{\dot{A}}\epsilon^{\dot{A}\dot{B}}\mathrm{l}^\star_{\dot{B}}, \delta^{\dot{A}}_A\epsilon^{AB}\mathrm{r}^\star_B) & (x_0,-\vec{x}) \\ (\mathrm{l}^A,\mathrm{r}^{\dot{A}}) & (x_0,\vec{x}) & \stackrel{\mathtt{T}}{\longleftrightarrow} & (\delta^{A\dot{A}}\mathrm{l}^\star_{\dot{A}}, \delta^{\dot{A}A}\mathrm{r}^\star_A) & (-x_0,\vec{x}) \end{array}$$

which is inducable on product representations.

5 The Standard Model Breakdown of P and CP

A relativistic dynamics, characterized by a Lagrangian for the fields involved, may be invariant with respect to an operation group G, e.g. the C, P and T reflections, or not. A breakdown of the symmetry can occur in two different ways: Either the symmetry is represented on the field value space V, but the Lagrangian is not G-invariant, or there does not even exist a G-representation on V. Both cases occur in the standard model for quark and lepton fields.

5.1 Standard Model Breakdown of P

The charge $\mathbf{U}(1)$ vertex in electrodynamics for a Dirac electron-positron field Ψ interacting with an electromagnetic gauge field Γ_j

$$-\Gamma_j\overline{\Psi}\gamma^j\Psi = -\Gamma_j(\mathrm{l}^\star\sigma^j\mathrm{l} + \mathrm{r}^\star\check{\sigma}^j\mathrm{r})$$

is invariant under P and T if the fields have the Weyl spinor induced behaviour given above.

In the standard model of leptons[5] with a left handed isospin doublet field L and a right handed isospin singlet field r the hypercharge $\mathbf{U}(1)$ and isospin $\mathbf{SU}(2)$ vertex with gauge fields A_j and $\vec{B}_j$ resp. and internal Pauli matrices $\vec{\tau}$ reads

$$-A_j(\mathrm{L}^\star\sigma^j\tfrac{\mathbf{1}_2}{2}\mathrm{L}+\mathrm{r}^\star\check{\sigma}^j\mathrm{r})+\vec{B}_j\mathrm{L}^\star\sigma^j\tfrac{\vec{\tau}}{2}\mathrm{L}$$

All gauge fields are assumed with the spinor induced reflection behaviour. The $\mathtt{P}$-invariance is broken in two different ways: One component of the lepton isodoublet, e.g. $\mathrm{l}=\frac{1-\tau_3}{2}\mathrm{L}\in W_L^-\cong\mathbb{C}^2$, can be used together with the right handed isosinglet r as a basis of a Dirac space $\Psi\in W_L^-\oplus W_R\cong\mathbb{C}^4$ with a representation of $\mathtt{P}$. This is impossible for the remaining unpaired left handed field $\frac{1+\tau_3}{2}\mathrm{L}\in W_L^+\cong\mathbb{C}^2$ - here $\mathtt{P}$ cannot even be defined. However, also for the left-right pair (l,r) the resulting gauge vertex breaks position space reflection $\mathtt{P}$ invariance via the familiar neutral weak interactions, induced by a vector field Z_j arising in addition to the $\mathbf{U}(1)$-electromagnetic gauge field Γ_j

$$\begin{array}{rl}-\frac{A_j+B_j^3}{2}\mathrm{l}^\star\sigma^j\mathrm{l}-A_j\mathrm{r}^\star\check{\sigma}^j\mathrm{r} & =-\Gamma_j\overline{\Psi}\gamma^j\Psi-Z_j\overline{\Psi}\gamma^j\gamma_5\Psi\\ \text{with} & \begin{pmatrix}\Gamma_j\\Z_j\end{pmatrix}=\frac{1}{4}\begin{pmatrix}3&1\\-1&1\end{pmatrix}\begin{pmatrix}A_j\\B_j^3\end{pmatrix}\end{array}$$

There is no parameter involved whose vanishing would lead to a $\mathtt{P}$-invariant dynamics.

5.2 $\mathtt{GP}$-Invariance in the Standard Model of Leptons

The $\mathtt{CP}$-reflection induced by the spinor ‘metric’

$$\begin{array}{ll}W_L\xleftrightarrow{\mathtt{CP}}W_R^T, & \mathrm{l}^A\leftrightarrow\delta^A_{\dot{A}}\epsilon^{\dot{A}\dot{B}}\mathrm{l}^\star_{\dot{B}}\\ W_R\xleftrightarrow{\mathtt{CP}}W_L^T, & \mathrm{r}^{\dot{A}}\leftrightarrow\delta^{\dot{A}}_A\epsilon^{AB}\mathrm{r}^\star_B\end{array}$$

has to include also a linear reflection of internal operation representations spaces in the case of Weyl spinors with nonabelian internal degrees of freedom.

For isospin $\mathbf{SU}(2)$-doublets this reflection is given by the Pauli isospinor reflection discussed above and is denoted as internal reflection by $\mathtt{I}=\epsilon$

$$\begin{array}{ccc}U & \overset{u}{\longrightarrow} & U\\ \epsilon\big\downarrow & & \big\downarrow\epsilon\\ U^T & \underset{\check{u}}{\longrightarrow} & U^T\end{array},\quad \begin{array}{l}u\in\mathbf{SU}(2)\ \text{(isospin)}\\ \psi^a\overset{\mathtt{I}}{\leftrightarrow}\epsilon^{ab}\psi^\star_b,\quad a,b=1,2\\ -\vec{\tau}=\epsilon^{-1}\circ\vec{\tau}^T\circ\epsilon\end{array}$$

Therewith the linear $\mathtt{GP}$-reflection as particle-antiparticle conjugation including nontrivial isopsin eigenvalues

$$\mathtt{G}=\mathtt{IC},\quad \mathtt{GP}=\mathtt{ICP}$$

reads for left handed Weyl spinors isospinors

$$W_L\otimes U\xleftrightarrow{\mathtt{GP}}W_R^T\otimes U^T,\quad \mathrm{L}^{Aa}\leftrightarrow\delta^A_{\dot{A}}\epsilon^{\dot{A}\dot{B}}\epsilon^{ab}\mathrm{L}^\star_{\dot{B}b}$$

The antilinear $\mathtt{T}$-reflection uses the $\mathbf{U}(2)$-scalar product

$$\begin{array}{l} U \stackrel{\star}{\longleftrightarrow} U^T, \quad U \times U \longrightarrow \mathbb{C}, \quad \langle \psi^a | \psi^b \rangle = \delta^{ab} \\ W_L \otimes U \stackrel{\mathtt{T}}{\longleftrightarrow} W_R^T \otimes U^T, \quad \mathrm{L}^{Aa} \leftrightarrow \delta^{A\dot{B}} \delta^{ab} \mathrm{L}^\star_{\dot{B}b} \end{array}$$

The isospin dual coincides with the time dual $U^T = U^\star$.

In the product $\mathtt{CPT}$ there arises - in the basis chosen - an isospin transformation $\epsilon^{ac}\delta_{cb} \cong e^{i\frac{\pi}{2}\tau_2} \in \mathbf{SU}(2)$

$$W_L \otimes U \stackrel{\mathtt{ICPT}}{\longleftrightarrow} W_L \otimes U, \quad \mathrm{L}^{Aa} \leftrightarrow \delta^{A\dot{B}} \delta_{\dot{B}B} \epsilon^{ac} \delta_{cb} \mathrm{L}^{Bb}$$

decisive to prove the $\mathtt{GPT}$-theorem with

$$\overline{\mathtt{ICPT}} \in \mathbf{SU}(2) \times \mathbf{SL}(\mathbb{C}^2)$$

With the spinor induced reflection behaviour for the gauge fields the standard model for leptons, i.e. with internal hupercharge-isospin action, allows the representation of $\mathtt{GP}$ and $\mathtt{T}$ with the gauge vertex above being $\mathtt{GP}$ and $\mathtt{T}$ invariant.

5.3 $\mathtt{CP}$-Problems for Quarks

If quark triplets and antitriplets which come with the dual defining $\mathbf{SU}(3)$-representations, are included in the standard model, an extended $\mathtt{CP}$-reflection has to employ a linear reflection γ between dual representation spaces of colour $\mathbf{SU}(3)$, i.e. an $\mathbf{SU}(3)$-invariant bilinear form of the representation space

$$\begin{array}{ccc} U & \stackrel{D(u)}{\longrightarrow} & U \\ \gamma \downarrow & & \downarrow \gamma \\ U^T & \underset{\check{D}(u)}{\longrightarrow} & U^T \end{array}, \quad \begin{array}{l} D: \mathbf{SU}(3) \longrightarrow \mathbf{GL}(U) \text{ (colour representation)} \\ \gamma^{-1} \circ D(u)^T \circ \gamma = D(u^{-1}) \text{ for all } u \in \mathbf{SU}(3) \end{array}$$

The situation for isospin $\mathbf{SU}(2)$ and colour $\mathbf{SU}(3)$ is completely different with respect to the existence of such a linear dual isomorphism γ: All irreducible $\mathbf{SU}(2)$-representations $[2T]$ with isospin $T = 0, \frac{1}{2}, 1, \ldots$ have an - up to a scalar factor - unique invariant bilinear form $\bigvee^{2T} \epsilon$ as product of the spinor 'metric', discussed above.

That is not the case for the colour representations. Some representations are linearly selfdual, some are not.

The complex irreducible representations of $\mathbf{SU}(3)$ are characterized by $[N_1, N_2]$ with two integers $N_{1,2} = 0, 1, 2, \ldots$. They arise from the two fundamental triplet representations, dual to each other and parametrizable with eight Gell-Mann matrices $\vec{\lambda}$

$$\begin{array}{l} \text{triplet: } [1,0] = u = e^{i\vec{\gamma}\vec{\lambda}}, \quad \text{antitriplet: } [0,1] = \check{u} = u^{-1T} = (e^{-i\vec{\gamma}\vec{\lambda}})^T \\ [N_1, N_2] \text{ acting on vector space } U \text{ with } \dim_{\mathbb{C}} U = \frac{(N_1+1)(N_2+1)(N_1+N_2+2)}{2} \end{array}$$

Dual representations have reflected integer values $[N_1, N_2] \leftrightarrow [N_2, N_1]$. Only those $\mathbf{SU}(3)$-representations whose weight diagram is central reflection symmetric in the real 2-dimensional weight vector space (appendix) have one, and only one, $\mathbf{SU}(3)$-invariant bilinear form[1], i.e. they are linearly selfdual. Dual representations have weights which are reflected to each other

$$\mathbf{weights}\,[N_1, N_2] \xleftrightarrow{-\mathbf{1}_2} \mathbf{weights}\,[N_2, N_1]$$

Therefore, one obtains as selfdual irreducible $\mathbf{SU}(3)$-representations

$$\begin{array}{rcl} \mathbf{weights}\,[N_1, N_2] = -\mathbf{weights}\,[N_1, N_2] & \iff & N_1 = N_2 = N \\ & \Rightarrow & \dim_{\mathbb{C}} U = (N+1)^3 = 1, 8, 27, \ldots \end{array}$$

E.g. for the octet $[1, 1]$ as adjoint $\mathbf{SU}(3)$-representation, the Killing form defines its selfduality.

A general remark (appendix): The Lie group $\mathbf{SL}(\mathbb{C}^{r+1})$ with its maximal compact subgroup $\mathbf{SU}(r+1)$ of rank r is defined as invariance group of the $\mathbb{C}^{r+1}$-volume elements which are totally antisymmetric $(r+1)$-linear forms $\epsilon^{a_1 \ldots a_{r+1}}$. Their complex finite dimensional irreducible representations are characterized by r integers $[N_1, \ldots, N_r]$ with the dual representations having the reflected order $[N_r, \ldots, N_1]$. The weights (eigenvalues) for dual representations are related to each other by the central weight space reflection $-\mathbf{1}_r$ which defines the linear particle-antiparticle conjugation $\mathtt{I}$ for $\mathbf{SU}(n)$. Only for $n = 2$ (isospin $\mathbf{SU}(2)$) all representations $[N = 2T]$ are selfdual with their invariant bilinear form arising from ϵ^{ab} for $[1]$. The $n = 2$ selfduality of the doublet $u(2) \cong \check{u}(2)$ is replaced for $n = 3$ by the equivalence of antisymmetric triplet square and antitriplet representation $u(3) \wedge u(3) \cong \check{u}(3)$, i.e. $3 \wedge 3 \cong \overline{3}$, with the obvious generalization $\bigwedge^{r} u(r+1) \cong \check{u}(r+1)$ for general rank r.

Obviously all $\mathbf{SU}(r+1)$-representations have an invariant sesquilinear form, the $\mathbf{SU}(r+1)$ scalar product. However, this antilinear structure cannot define a linear particle-antiparticle conjugation.

It is impossible to define a $\mathtt{CP}$-extending duality induced linear $\mathtt{GP}$-reflection for the irreducible complex 3-dimensional quark representation spaces since there does not exist a colour $\mathbf{SU}(3)$-invariant bilinear form of the triplet space $U \cong \mathbb{C}^3$. Or equivalently: There does not exist a (3×3)-matrix γ for the reflection $-\vec{\lambda} = \gamma^{-1} \circ \vec{\lambda}^T \circ \gamma$ of all eight Gell-Mann matrices. Therewith there arise also problems to define an $\mathbf{SU}(3)$-compatible time reflection for quark triplet fields. Could all that be the reason for the breakdown of $\mathtt{CP}$-invariance coming on the quark field sector and its parametrization (e.g. Cabibbo-Kobayashi-Maskawa) with three families of colour triplets?

A Central Reflections of Lie Algebras

A representation of a group G on a vector space V is *selfdual* if it is equivalent to its dual representation, defined by the inversed transposed action on the linear forms V^T

$$\left.\begin{array}{l} D : G \longrightarrow \mathbf{GL}(V) \\ \check{D} : G \longrightarrow \mathbf{GL}(V^T) \end{array}\right\}, \quad \check{D}(g) = D(g^{-1})^T$$

i.e. if the following diagram with a linear or antilinear isomorphism $\zeta : V \longrightarrow V^T$ commutes with the action of all group elements

$$\begin{array}{ccccc} & V & \overset{D(g)}{\longrightarrow} & V & \\ \zeta & \downarrow & & \downarrow & \zeta\,, \\ & V^T & \underset{\check D(g)}{\longrightarrow} & V^T & \end{array} \qquad \zeta^{-1}\circ D(g)^T \circ \zeta = D(g^{-1}) \text{ for all } g \in G$$

Selfduality is equivalent to the existence of a nondegenerate bilinear (for linear ζ) or sesquilinear form (for antilinear ζ) of the vector space V

$$\begin{array}{rl} V \times V \longrightarrow \mathbb{C}, & \zeta(w, v) = \langle \zeta(w), v\rangle \\ \text{selfdual} & \zeta(g \bullet w, g \bullet v) = \zeta(w, v), \quad g \bullet v = D(g).v \end{array}$$

For the Lie algebra $L = \log G$ of a Lie group G with dual representations in the endomorphism algebras $\mathbf{AL}(V)$ and $\mathbf{AL}(V^T)$ which are negative transposed to each other

$$\left.\begin{array}{l} \mathcal{D} : L \longrightarrow \mathbf{AL}(V) \\ \check{\mathcal{D}} : L \longrightarrow \mathbf{AL}(V^T) \end{array}\right\}, \quad \check{\mathcal{D}}(l) = -\mathcal{D}(l)^T$$

a selfduality isomorphism, i.e. the reflection $V \overset{\zeta}{\longleftrightarrow} V^T$ fulfills

$$\zeta(l \bullet w, v) = -\zeta(w, l \bullet v), \quad l \bullet v = \mathcal{D}(l).v$$

and defines the *central reflection of the Lie algebra* in the representation

$$\begin{array}{ccccc} & V & \overset{\mathcal{D}(l)}{\longrightarrow} & V & \\ \zeta & \downarrow & & \downarrow & \zeta\,, \\ & V^T & \underset{\check{\mathcal{D}}(l)}{\longrightarrow} & V^T & \end{array} \quad \zeta^{-1}\circ \mathcal{D}(l)^T \circ \zeta = -\mathcal{D}(l) \text{ for all } l \in \log G$$

With Schur's lemma, an irreducible complex finite dimensional representation of a group or Lie algebra can have at most - up to a constant - one invariant bilinear and one invariant sesquilinear form. E.g. Pauli spinors for $\mathbf{SU}(2)$ have both, ϵ^{AB} (bilinear) and δ^{AB} (sesquilinear, scalar product), $A, B = 1, 2$, quark triplets have only a scalar product δ^{ab}, $a, b = 1, 2, 3$, Weyl spinors for $\mathbf{SL}(\mathbb{C}^2)$ have only the bilinear 'metric' ϵ^{AB}.

For a simple Lie algebra L of rank r, the weights (eigenvalue vectors for a Cartan subalgebra) of dual representations $\mathcal{D}$ and $\check{\mathcal{D}}$ are related to each other by the central reflection of the weight vector space $\mathbb{R}^r$

$$\mathbf{weights}\,\mathcal{D}[L] \overset{-\mathbf{1}_r}{\longleftrightarrow} \mathbf{weights}\,\check{\mathcal{D}}[L]$$

which may be induced by a linear isomorphism ζ of the dual representation spaces. Therewith: Such a linear isomorphism for an L-representation exists[1] if, and only if, the weights of the representation $\mathcal{D} : L \longrightarrow \mathbf{AL}(V)$ are invariant under central reflection

$$V \xleftrightarrow{\zeta} V^T \iff \mathbf{weights}\,\mathcal{D}[L] = -\mathbf{weights}\,\mathcal{D}[L]$$

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